

MONOPOLE-ANTIMONOPOLE SOLUTIONS OF EINSTEIN-YANG-MILLS-HIGGS THEORY

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Abstract

We construct static axially symmetric solutions of $SU(2)$ Einstein-Yang-Mills-Higgs theory in the topologically trivial sector, representing gravitating monopole-antimonopole pairs, linked to the Bartnik-McKinnon solutions.

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1 Introduction

SU(2) Yang-Mills-Higgs (YMH) theory possesses monopole [1], multimonopole [2, 3, 4], and monopole-antimonopole pair solutions [5, 6]. The magnetic charge of these solutions is proportional to their topological charge. While monopole and multimonopole solutions reside in topologically non-trivial sectors, the monopole-antimonopole pair solution is topologically trivial.

When gravity is coupled to YMH theory, a branch of gravitating monopole solutions emerges smoothly from the monopole solution of flat space [7, 8, 9]. The coupling constant α , entering the Einstein-Yang-Mills-Higgs (EYMH) equations, is proportional to the gravitational constant G and to the square of the Higgs vacuum expectation value η . The monopole branch ends at a critical value α_{cr} , beyond which gravity becomes too strong for regular monopole solutions to persist, and collapse to charged black holes is expected [7, 8, 9]. Indeed, when the critical value α_{cr} is reached, the gravitating monopole solutions develop a degenerate horizon [10], and the exterior space time of the solution corresponds to the one of an extremal Reissner-Nordström (RN) black hole with unit magnetic charge [7, 8, 9, 11].

Beside the fundamental gravitating monopole solution, EYMH theory possesses radially excited monopole solutions, not present in flat space [7, 8, 9]. These excited solutions also develop a degenerate horizon at some critical value of the coupling constant, but they shrink to zero size in the limit $\alpha \rightarrow 0$. Rescaling of the solutions reveals, that in this limit the Bartnik-McKinnon (BM) solutions [12] of Einstein-Yang-Mills (EYM) theory are recovered. For the excited solutions the limit $\alpha \rightarrow 0$ therefore corresponds to the limit of vanishing Higgs expectation value, $\eta \rightarrow 0$.

In this letter we investigate how gravity affects the static axially symmetric monopole-antimonopole pair (MAP) solution of flat space [6], and we elucidate, that curved space supports a rich spectrum of MAP solutions, not present in flat space.

In particular, we show that, with increasing α , a branch of gravitating MAP solutions emerges smoothly from the flat space MAP solution, and ends at a critical value $\alpha_{\text{cr}}^{(1)}$, when gravity becomes too strong for regular MAP solutions to persist. But while the branch of monopole solutions can merge into an extremal RN black hole solution at the critical α , there seems to be no neutral black hole solution with degenerate horizon available for the MAP solutions to merge into. Indeed we find that at $\alpha_{\text{cr}}^{(1)}$ a second branch of MAP solutions emerges, extending back to $\alpha = 0$. Along this upper branch the MAP solutions shrink to zero size, in the limit $\alpha \rightarrow 0$, and approach the BM solution with one node (after rescaling).

Since the BM solution with one node is related to a branch of MAP solutions, it immediately suggests itself that the excited BM solutions with k nodes are related to branches of excited MAP solutions. Indeed, constructing the first excited MAP solution by starting from the BM solution with two nodes, we find, that it represents a MAP

solution, possessing two monopole-antimonopole pairs.

2 Axially symmetric ansatz

The static axially symmetric MAP solutions of SU(2) EYMH theory with action

$$S = \int \left(\frac{R}{16\pi G} - \frac{1}{2e} \text{Tr}(F_{\mu\nu} F^{\mu\nu}) - \frac{1}{4} \text{Tr}(D_\mu \Phi D^\mu \Phi) \right) \sqrt{-g} d^4x \quad (1)$$

(with Yang-Mills coupling constant e , and vanishing Higgs self-coupling), are obtained in isotropic coordinates with metric [13]

$$ds^2 = -f dt^2 + \frac{m}{f} (dr^2 + r^2 d\theta^2) + \frac{l}{f} r^2 \sin^2 \theta d\varphi^2, \quad (2)$$

where f , m and l are only functions of r and θ . The MAP ansatz reads for the purely magnetic gauge field ($A_0 = 0$) [6]

$$A_\mu dx^\mu = \frac{1}{2e} \left\{ \left(\frac{H_1}{r} dr + 2(1 - H_2) d\theta \right) \tau_\varphi - 2 \sin \theta \left(H_3 \tau_r^{(2)} + (1 - H_4) \tau_\theta^{(2)} \right) d\varphi \right\} \quad (3)$$

and for the Higgs field

$$\Phi = \left(\Phi_1 \tau_r^{(2)} + \Phi_2 \tau_\theta^{(2)} \right), \quad (4)$$

with $su(2)$ matrices (composed of the standard Pauli matrices τ_i)

$$\begin{aligned} \tau_r^{(2)} &= \sin 2\theta \tau_\rho + \cos 2\theta \tau_3, & \tau_\theta^{(2)} &= \cos 2\theta \tau_\rho - \sin 2\theta \tau_3, \\ \tau_\rho &= \cos \varphi \tau_1 + \sin \varphi \tau_2, & \tau_\varphi &= -\sin \varphi \tau_1 + \cos \varphi \tau_2. \end{aligned} \quad (5)$$

The four gauge field functions H_i and the two Higgs field functions Φ_i depend only on r and θ . We fix the residual gauge degree of freedom [3, 13, 6] by choosing the gauge condition $r \partial_r H_1 - 2 \partial_\theta H_2 = 0$ [6].

To obtain regular asymptotically flat solutions with finite energy density we impose at the origin ($r = 0$) the boundary conditions

$$\begin{aligned} H_1 &= H_3 = H_2 - 1 = H_4 - 1 = 0, \\ \sin 2\theta \Phi_1 + \cos 2\theta \Phi_2 &= 0, & \partial_r (\cos 2\theta \Phi_1 - \sin 2\theta \Phi_2) &= 0, \\ \partial_r f &= \partial_r m = \partial_r l = 0. \end{aligned}$$

On the z -axis the functions H_1, H_3, Φ_2 and the derivatives $\partial_\theta H_2, \partial_\theta H_4, \partial_\theta \Phi_1, \partial_\theta f, \partial_\theta m, \partial_\theta l$ have to vanish, while on the ρ -axis the functions $H_1, 1 - H_4, \Phi_2$ and the derivatives

$\partial_\theta H_2, \partial_\theta H_3, \partial_\theta \Phi_1, \partial_\theta f, \partial_\theta m, \partial_\theta l$ have to vanish. For solutions with vanishing net magnetic charge the gauge potential approaches a pure gauge at infinity. The corresponding boundary conditions for the fundamental MAP solution are given by [6]

$$H_1 = H_2 = 0, H_3 = \sin \theta, 1 - H_4 = \cos \theta, \Phi_1 = \eta, \Phi_2 = 0, f = m = l = 1. \quad (6)$$

Introducing the dimensionless coordinate $x = r\eta e$ and the Higgs field $\phi = \Phi/\eta$, the equations depend only on the coupling constant α , $\alpha^2 = 4\pi G\eta^2$. The mass M of the MAP solutions can be obtained directly from the total energy-momentum “tensor” $\tau^{\mu\nu}$ of matter and gravitation, $M = \int \tau^{00} d^3r$ [14], or equivalently from $M = -\int (2T_0^0 - T_\mu^\mu) \sqrt{-g} dr d\theta d\phi$, yielding the dimensionless mass $\mu = \frac{4\pi\eta}{e} M$.

3 Solutions

Subject to the above boundary conditions, we solve the equations numerically [15]. In the limit $\alpha \rightarrow 0$, the lower branch of gravitating MAP solutions emerges smoothly from the flat space solution [6]. The modulus of the Higgs field of these MAP solutions possesses two zeros, $\pm z_0$, on the z -axis, corresponding to the location of the monopole and antimonopole, respectively.

With increasing α the monopole and antimonopole move closer to the origin, and the mass μ of the solutions decreases. The lower branch of MAP solutions ends at the critical value $\alpha_{\text{cr}}^{(1)} = 0.670$. In Fig. 1 we show the energy density $\varepsilon = -T_0^0 = -L_M$ of the MAP solution at $\alpha_{\text{cr}}^{(1)}$. It possesses maxima on the positive and negative z -axis close to the locations of the monopole and antimonopole and a saddle point at the origin.

Forming a second branch, the MAP solutions evolve smoothly backwards from $\alpha_{\text{cr}}^{(1)}$ to $\alpha = 0$. In the limit $\alpha \rightarrow 0$ the mass μ diverges on this upper branch, and the locations of the monopole and antimonopole approach the origin, $\pm z_0 \rightarrow 0$, as seen in Fig. 2. At the same time the MAP solution shrinks to zero.

Rescaling the coordinate $x = \hat{x}\alpha$ and the Higgs field $\phi = \hat{\phi}/\alpha$ reveals that the axially symmetric MAP solutions approach the spherically symmetric $k = 1$ BM solution on the upper branch as $\alpha \rightarrow 0$. Consequently, also the scaled mass $\hat{\mu} = \alpha\mu$ of the MAP solutions tends to the mass of the $k = 1$ BM solution, as seen in Fig. 3. On the upper branch the limit $\alpha \rightarrow 0$ thus corresponds to the limit $\eta \rightarrow 0$ (with fixed G). We note that the ansatz (3) for the gauge potential includes the spherically symmetric BM ansatz,

$$H_1 = 0, \quad 1 - H_2 = \frac{1}{2}(1 - w), \quad H_3 = \frac{1}{2}\sin\theta(1 - w), \quad 1 - H_4 = \frac{1}{2}\cos\theta(1 - w), \quad (7)$$

where w denotes the gauge field function of the BM solution.

Anticipating the existence of excited MAP solutions, linked to the BM solutions with k nodes on their upper branches, we construct the first excited MAP solution,

starting from the $k = 2$ BM solution. Since the boundary conditions of the $k = 2$ BM solution differ from those of the $k = 1$ BM solution at infinity, the boundary conditions of the first excited MAP solution at infinity must be modified accordingly,

$$H_1 = H_3 = 0, \quad H_2 = H_4 = 1, \quad \phi_1 = \pm \cos 2\theta, \quad \phi_2 = \mp \sin 2\theta, \quad f = m = l = 1. \quad (8)$$

The upper branch of the first excited MAP solutions ends at the critical value $\alpha_{\text{cr}}^{(2)} = 0.128$, from where the lower branch of the excited MAP solutions evolves smoothly backwards to $\alpha = 0$. As seen in Fig.3, in the limit $\alpha \rightarrow 0$ the scaled mass $\hat{\mu}$ approaches the mass of the $k = 2$ BM solution on the upper branch, and the mass of the $k = 1$ BM solution on the lower branch.

The modulus of the Higgs field of the first excited MAP solution possesses four zeros, $\pm z_0^+$ and $\pm z_0^-$, located on the z -axis, representing two monopole-antimonopole pairs. The locations of the monopole and antimonopole on the positive z -axis, z_0^+ resp. z_0^- , are shown in Fig. 2 as functions of α , together with the node z_0 of the fundamental MAP solution. As $\alpha \rightarrow 0$, z_0^- tends to zero on both branches; in contrast, z_0^+ tends to zero only on the upper branch. On the lower branch z_0^+ tends to z_0 , the location of the monopole of the fundamental MAP solution.

Inspecting the limit $\alpha \rightarrow 0$ for the first excited MAP solution on the lower branch reveals, that in terms of the radial coordinate $x = r\eta e$, the solution differs from the fundamental MAP solution on its lower branch only near the origin, where the excited MAP solution develops a discontinuity. In terms of the coordinate $\hat{x} = x/\alpha$, on the other hand, the first excited MAP solution approaches the $k = 1$ BM solution for all values of \hat{x} , except at infinity. Hence, the first excited MAP solution does not possess a counterpart in flat space.

4 Conclusions

Having constructed the fundamental and the first excited MAP solutions, we expect, that EYM theory possesses a whole sequence of MAP solutions, labeled by the number of monopole-antimonopole pairs k . Each MAP solution forms two branches, merging and ending at $\alpha_{\text{cr}}^{(k)}$. In the limit $\alpha \rightarrow 0$, the upper branch of the k th MAP solution always reaches the Bartnik-McKinnon solution with k nodes, while the lower branch of the k th MAP solution always reaches the Bartnik-McKinnon solution with $k - 1$ nodes, except for $k = 1$, where the flat space MAP solution is reached in the limit $\alpha \rightarrow 0$. We conjecture, that the critical values $\alpha_{\text{cr}}^{(k)}$ decrease with k , such that, as a function of α , the scaled mass $\hat{\mu}$ assumes a characteristic “Christmas tree” shape. Thus instead of the single MAP solution present in flat space, in curved space a whole tower of MAP solutions appears. An analogous pattern is encountered for gravitating Skyrmions, which are likewise linked to the BM solutions [16]. We expect the gravitating MAP solutions to be unstable like the flat space MAP solution [5].

For the gravitating monopole solutions a regular event horizon can be imposed [7, 8, 9], yielding magnetically charged black hole solutions with hair. Likewise for the MAP solutions of EYMH theory a regular event horizon can be imposed, yielding static axially symmetric and neutral black hole solutions with hair [17]. Within the framework of distorted isolated horizons the masses of these black hole solutions may possibly be simply related to the masses of the corresponding regular solutions [18].

It is interesting, that the spherically symmetric BM solutions of EYM theory appear in the limit $\alpha \rightarrow 0$ of the axially symmetric MAP solutions. But EYM theory also possesses static axially symmetric regular solutions, which are not spherically symmetric [13]. Could these solutions also appear in the $\alpha \rightarrow 0$ limit of more general [19] gravitating MAP solutions? We conjecture, that EYMH theory allows for the existence of MAP solutions, consisting of pairs of static axially symmetric multimonopoles, where each multimonopole has winding number n [2, 3]. It is then conceivable that such multimonopole-antimultimonopole solutions will form an analogous set of solutions as the ones encountered above, but with their upper branches reaching axially symmetric EYM solutions with winding number n in the $\alpha \rightarrow 0$ limit.

But also flat space should contain further interesting solutions, for instance an antimonopole-monopole-antimonopole system, with the poles located symmetrically with respect to the origin on the z -axis.

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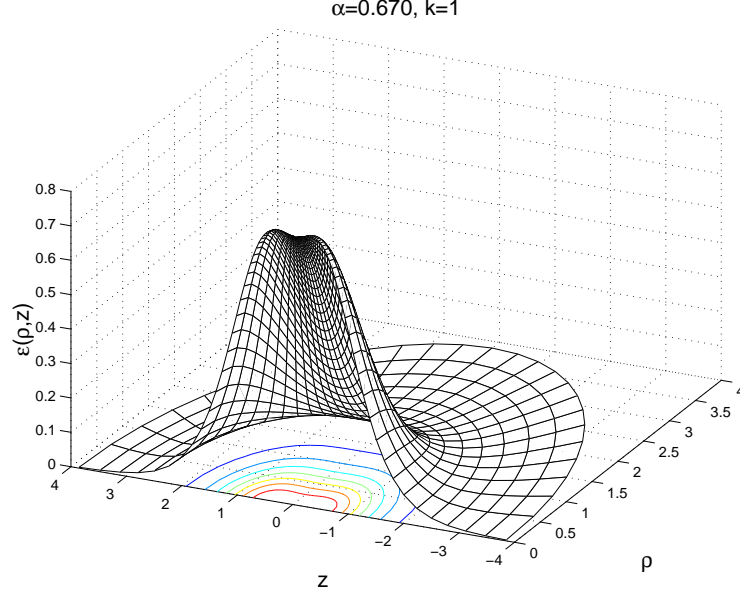


Figure 1: The energy density $\varepsilon(\rho, z)$ is shown for the fundamental MAP solution at $\alpha_{\text{cr}}^{(1)} = 0.67$.

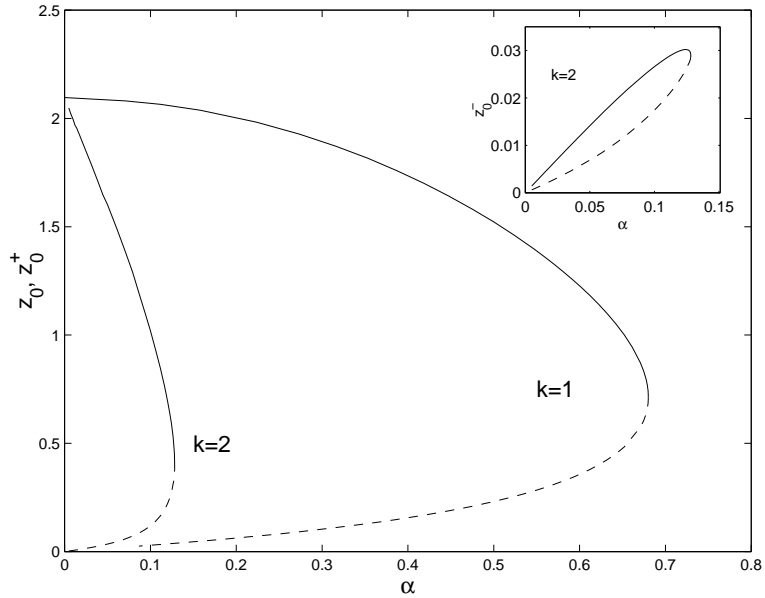


Figure 2: For the fundamental ($k = 1$) and the first excited ($k = 2$) MAP solution the locations of the monopole, z_0 resp. z_0^+ , are shown as functions of α . In the inset the location of the antimonopole, z_0^- , of the first excited MAP solution is shown. The solid and dashed lines correspond to the lower and upper (mass) branches, respectively.

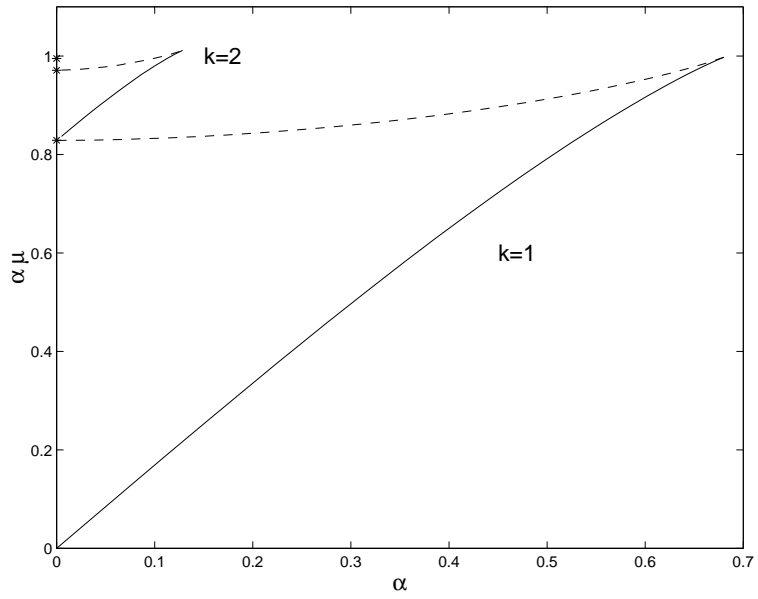


Figure 3: The scaled mass $\hat{\mu} = \alpha\mu$ is shown as a function of α for the fundamental ($k = 1$) and the first excited ($k = 2$) MAP solution. The solid and dashed lines correspond to the lower and upper (mass) branches, respectively. The stars indicate the masses of the $k = 1, 2, 3$ (from bottom to top) BM solutions.